

GAUGE THEORY ON PROJECTIVE SURFACES AND ANTI-SELF-DUAL EINSTEIN METRICS IN DIMENSION FOUR

MACIEJ DUNAJSKI AND THOMAS METTLER

ABSTRACT. Given a projective structure on a surface N , we show how to canonically construct a neutral signature Einstein metric with non-zero scalar curvature on the total space M of a certain rank 2 affine bundle $M \rightarrow N$. The resulting Einstein metric has anti-self-dual conformal curvature and admits a parallel field of anti-self-dual planes. We show that Einstein metrics arising from this construction admit smooth deformations to Ricci-flat metrics if the corresponding projective structures admit a representative connection with totally skew-symmetric Ricci tensor. The homogeneous Einstein metric corresponding to the flat projective structure on \mathbb{RP}^2 is the non-compact real form of the Fubini-Study metric on $M = \mathrm{SL}(3, \mathbb{R})/\mathrm{GL}(2, \mathbb{R})$.

Our construction is a combination of the Cartan and tractor bundle approaches to projective differential geometry with a projectively invariant gauge theory of a connection and a pair of Higgs fields developed by Calderbank.

1. INTRODUCTION

The aim of this paper is to canonically associate a neutral signature Einstein metric to any two-dimensional projective structure. Recall that a projective structure $[\nabla]$ on a surface N is an equivalence class of affine torsion-free connections on TN which share the same unparametrised geodesics. We shall provide two different constructions of this metric: one as a $\mathrm{GL}(2, \mathbb{R})$ quotient of the total space of the Cartan bundle associated to $(N, [\nabla])$, and another one as a solution to Calderbank's projective pair equations [7] corresponding to the tractor connection on a rank-3 vector bundle E over N . The Einstein metric is defined on the total space of a rank-2 affine bundle $M \rightarrow N$ which is the complement of an \mathbb{RP}^1 sub-bundle of the projectivised tractor bundle $\mathbb{P}(E) \rightarrow N$ of $(N, [\nabla])$. If a local trivialisation is chosen the metric takes the form

$$g = (d\xi_i - (\Gamma_{ij}^k \xi_k - \Lambda \xi_i \xi_j - \Lambda^{-1} P_{ji}) dx^j) \odot dx^i \quad (1.1)$$

where the indices i, j, k take values from 1 to 2, the coordinates $x = (x^i)$ refer to a local chart U in N , the coordinates $\xi = (\xi_i)$ are defined on an affine chart of the fibres of $\mathbb{P}(E)$, Γ_{ik}^j are the Christoffel symbols of a $[\nabla]$ -representative connection, P_{ij} is the Schouten tensor of Γ_{ik}^j , given in terms of the Ricci tensor R_{ij} by $P_{ij} = R_{(ij)} + \frac{1}{3}R_{[ij]}$ and $\Lambda = 1$. We shall show (Theorem 5.1) that the formula (1.1) with $\Lambda = 1$ defines a metric which does not depend on the choice of a connection in the projective class.

Date: September 15, 2015.

All metrics in the one-parameter family (1.1) with arbitrary non-zero Λ have anti-self-dual (ASD) Weyl tensor, and are Einstein with non-zero scalar curvature equal to -24Λ . Moreover (Theorem 5.3) all ASD Einstein metrics which admit a parallel ASD totally null distributions are locally of the form (1.1). In Theorem 6.3 we shall establish a one-to-one correspondence between projective vector fields on $(N, [\nabla])$ and Killing vectors on (M, g) which additionally preserve a canonically defined symplectic structure on M given by

$$\Omega = d\xi_i \wedge dx^i + \frac{1}{\Lambda} P_{ij} dx^i \wedge dx^j. \quad (1.2)$$

This symplectic structure is parallel w.r.t the Levi-Civita connection of g only iff the projective Cotton tensor of $(N, [\nabla])$ vanishes. In Theorem 3.3 we shall construct the pair (g, Ω) for the choice $\Lambda = 1$ as a bi-Lagrangian structure on the quotient of the Cartan bundle of $(N, [\nabla])$ by $\text{GL}(2, \mathbb{R})$.

The flat projective structure on $N = \mathbb{RP}^2$ gives rise to a non-flat homogeneous metric on $M = \text{SL}(3, \mathbb{R})/\text{GL}(2, \mathbb{R})$ which is a neutral signature analogue of the Fubini-Study metric. In this case the Schouten tensor vanishes, and g admits a flat limit where $\Lambda = 0$. Motivated by this example we shall find all projective structures where the limit $\Lambda \rightarrow 0$ exists. These projective structures are characterised by the existence of a connection with totally skew-symmetric Ricci tensor, and the resulting metrics (1.1) with $\Lambda = 0$ are Ricci-flat and hence hyper-symplectic. These metrics are given by formula (7.2).

This paper mainly concerns itself with the two-dimensional case, but there are obvious higher dimensional generalisations which we briefly discuss in an Appendix.

Acknowledgments. The authors wish to thank Andreas Čap, Andrzej Derdziński and Nigel Hitchin for helpful discussions regarding the contents of this paper. TM is grateful for travel support via the grant SNF 200020_144438 of the Swiss National Science Foundation.

2. PROJECTIVE DIFFERENTIAL GEOMETRY AND ANTI-SELF-DUALITY

In this preliminary Section we shall summarise known facts about projective structures on a surface, and neutral signature conformal structures in four dimensions which underlie the results of the paper.

2.1. Algebraic preliminaries. As usual, we let \mathbb{R}^n denote the space of column vectors of height n with real entries and \mathbb{R}_n the space of row vectors of length n with real entries. Matrix multiplication $\mathbb{R}_n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a non-degenerate pairing identifying \mathbb{R}_n with the dual vector space of \mathbb{R}^n .

Let $\mathbb{RP}^2 = (\mathbb{R}^3 \setminus \{0\})/\mathbb{R}^*$ denote space of lines in \mathbb{R}^3 through the origin, i.e., two-dimensional real projective space. For any nonzero $x \in \mathbb{R}^3$ let $[x]$ denote its corresponding point in \mathbb{RP}^2 . Let $\mathbb{RP}_2 = (\mathbb{R}_3 \setminus \{0\})/\mathbb{R}^*$ denote the dual projective space and likewise for any nonzero $\xi \in \mathbb{R}_3$ we denote by $[\xi]$ its corresponding point in \mathbb{RP}_2 .

The group $\mathrm{SL}(3, \mathbb{R})$ acts from the left on \mathbb{R}^3 by matrix multiplication and this action descends to define a transitive left action on \mathbb{RP}^2 . Likewise, $\mathrm{SL}(3, \mathbb{R})$ acts on \mathbb{R}_3 from the left by the rule

$$h \cdot \xi = \xi h^{-1}$$

for $h \in \mathrm{SL}(3, \mathbb{R})$ and this action descends to define a transitive left action on \mathbb{RP}_2 . The stabiliser subgroup of $[x_0] \in \mathbb{RP}^2$ where $x_0 = {}^t(1 \ 0 \ 0)$ will be denoted by G , so that $\mathbb{RP}^2 \simeq \mathrm{SL}(3, \mathbb{R})/G$. The elements of $G \subset \mathrm{SL}(3, \mathbb{R})$ are matrices of the form

$$b \rtimes a = \begin{pmatrix} \det a^{-1} & b \\ 0 & a \end{pmatrix},$$

with $a \in \mathrm{GL}(2, \mathbb{R})$ and $b \in \mathbb{R}_2$. Denoting by $\mathbb{RP}_1 \subset \mathbb{RP}_2$ the projective line consisting of those elements $[\xi] \in \mathbb{RP}_2$ which satisfy $[\xi] \cdot [x_0] = 0$, the group G acts faithfully from the left by affine transformations on the affine 2-space $\mathbb{A}_2 = \mathbb{RP}_2 \setminus \mathbb{RP}_1$. Indeed, if we represent an element in \mathbb{A}_2 by a vector $(1, \xi) \in \mathbb{R}_3$ with $\xi \in \mathbb{R}_2$, we obtain

$$(1, \xi) \begin{pmatrix} \det a^{-1} & b \\ 0 & a \end{pmatrix}^{-1} = (\det a, -ba^{-1} \det a + \xi a^{-1})$$

so that the induced affine transformation is

$$(b \rtimes a) \cdot \xi = \xi a^{-1} \det a^{-1} - ba^{-1}.$$

Consequently, we may naturally think of G as the 2-dimensional real affine group.

2.2. Projective differential geometry. Let N be a connected smooth surface. A projective structure $[\nabla]$ on N is an equivalence class of torsion-free connections on TN which share the same unparametrised geodesics. Two connections ∇ and $\hat{\nabla}$ belong to the same projective structure if their geodesic sprays have the same projections to the projectivised tangent bundle $\mathbb{P}(TN)$ or equivalently, if there exists a one-form Υ on N such that

$$\hat{\Gamma}_{jk}^i = \Gamma_{jk}^i + \delta_j^i \Upsilon_k + \delta_k^i \Upsilon_j. \quad (2.1)$$

Given a connection $\nabla \in [\nabla]$, its curvature is defined by

$$[\nabla_i, \nabla_j]X^k = R_{ij}{}^k{}_l X^l.$$

It can be uniquely decomposed as

$$R_{ij}{}^k{}_l = \delta_i^k P_{jl} - \delta_j^k P_{il} - 2P_{[ij]} \delta_l^k, \quad (2.2)$$

where P_{ij} is the Schouten tensor. In dimension higher than two there is an additional (Weyl) term in this decomposition which vanishes identically in the surface case. If we change the connection in the projective class using (2.1) then

$$\hat{P}_{ij} = P_{ij} - \nabla_i \Upsilon_j + \Upsilon_i \Upsilon_j, \quad \hat{P}_{[ij]} = P_{[ij]} - \nabla_{[i} \Upsilon_{j]}. \quad (2.3)$$

For a non-zero real number w let $\mathcal{E}(w) := (\Lambda^2(T^*N))^{-w/3}$ be the line bundle of projective densities of weight w .¹ The torsion-free connection ∇ on TN induces a connection on $\mathcal{E}(w)$, which – by standard abuse of notation – we denote by ∇ as well. Under the projective change (2.1) we obtain

$$\hat{\nabla}_i s = \nabla_i s + w \Upsilon_i s,$$

where s is any section of $\mathcal{E}(w)$.

If N is orientable, we may restrict attention to connections in $[\nabla]$ which preserve some fixed volume form $\epsilon \in \Gamma(\Lambda^2(T^*N))$, so that $\nabla_i \epsilon_{jk} = 0$. We shall refer to such connections as *special* [19]. Note that special connections always exist globally. For special connections the Schouten tensor is symmetric, that is $P_{[ij]} = 0$. The residual freedom in special connections within a given projective class is given by (2.1) where $\Upsilon_i = \nabla_i f$ is a gradient of an arbitrary smooth function $f : N \rightarrow \mathbb{R}$. The special condition is preserved if $\hat{\epsilon}_{ij} = e^{3f} \epsilon_{ij}$.

2.3. The Cartan geometry of a projective surface. In [10], Cartan associates to a projective structure $[\nabla]$ on a smooth surface N a Cartan geometry $(\pi : B_{[\nabla]} \rightarrow N, \theta)$ of type $(\mathrm{SL}(3, \mathbb{R}), G)$ which consists of a principal right G -bundle $\pi : B_{[\nabla]} \rightarrow N$ together with a Cartan connection $\theta \in \Omega^1(B_{[\nabla]}, \mathfrak{sl}(3, \mathbb{R}))$ having the following properties:

- (i) $\theta(X_v) = v$ for all fundamental vector fields X_v on $B_{[\nabla]}$;
- (ii) $\theta_u : T_u B_{[\nabla]} \rightarrow \mathfrak{sl}(3, \mathbb{R})$ is an isomorphism for all $u \in B_{[\nabla]}$;
- (iii) $R_g^* \theta = \mathrm{Ad}(g^{-1}) \theta = g^{-1} \theta g$ for all $g \in G$;
- (iv) Writing

$$\theta = \begin{pmatrix} -\mathrm{tr} \phi & \eta \\ \omega & \phi \end{pmatrix}$$

for an \mathbb{R}^2 -valued 1-form $\omega = (\omega^i)$, an \mathbb{R}_2 -valued 1-form $\eta = (\eta_i)$ and a $\mathfrak{gl}(2, \mathbb{R})$ -valued 1-form $\phi = (\phi_j^i)$, the leaves of the foliation defined by the equations $\omega^2 = \phi_1^2 = 0$ project to N to be the geodesics of $[\nabla]$;

- (v) The curvature 2-form Θ satisfies

$$\Theta = d\theta + \theta \wedge \theta = \begin{pmatrix} 0 & L(\omega \wedge \omega) \\ 0 & 0 \end{pmatrix}, \quad (2.4)$$

for a smooth curvature function $L : B_{[\nabla]} \rightarrow \mathrm{Hom}(\mathbb{R}^2 \wedge \mathbb{R}^2, \mathbb{R}_2)$.

Note the Bianchi-identity

$$d\Theta = \Theta \wedge \theta - \theta \wedge \Theta,$$

the algebraic part of which reads

$$0 = L(\omega \wedge \omega) \wedge \omega. \quad (2.5)$$

¹More precisely, by this we mean the bundle associated to the coframe bundle of N via the $\mathrm{GL}(2, \mathbb{R})$ representation $a \mapsto |\det a|^{w/3}$.

A projective structure $[\nabla]$ is called *flat* if locally $[\nabla]$ is defined by a flat connection. A consequence of Cartan's construction is that a projective structure is flat if and only if L vanishes identically.

Remark 2.1. Cartan's bundle is unique in the following sense: If $(\hat{\pi} : \hat{B}_{[\nabla]} \rightarrow N, \hat{\theta})$ is another Cartan geometry of type $(\mathrm{SL}(3, \mathbb{R}), G)$ satisfying the properties (iii), (iv), (v), then there exists a G -bundle isomorphism $\psi : B_{[\nabla]} \rightarrow \hat{B}_{[\nabla]}$ so that $\psi^* \hat{\theta} = \theta$.

Fixing a representative connection ∇ in the projective equivalence class $[\nabla]$ allows to give an explicit description of the Cartan geometry $(\pi : B_{[\nabla]} \rightarrow N, \theta)$. Let $v : F \rightarrow N$ denote the coframe bundle of N whose fibre at a point $p \in N$ consists of the linear isomorphisms $u : T_p N \rightarrow \mathbb{R}^2$. The group $\mathrm{GL}(2, \mathbb{R})$ acts transitively from the right on each v -fibre by the rule $R_a(u) = u \cdot a = a^{-1} \circ u$ for all $a \in \mathrm{GL}(2, \mathbb{R})$. This action turns $v : F \rightarrow N$ into a principal right $\mathrm{GL}(2, \mathbb{R})$ -bundle. The bundle $F \rightarrow N$ is equipped with a tautological \mathbb{R}^2 -valued 1-form $\omega = (\omega^i)$ satisfying the equivariance property $(R_a)^* \omega = a^{-1} \omega$, where the 1-form ω is defined by $\omega_u = u \circ v'_u$.

Suppose $\varphi = (\varphi_j^i) \in \Omega^1(F, \mathfrak{gl}(2, \mathbb{R}))$ is the connection 1-form of $\nabla \in [\nabla]$, then we have the structure equations

$$\begin{aligned} d\omega^i &= -\varphi_j^i \wedge \omega^j, \\ d\varphi_l^k + \varphi_j^k \wedge \varphi_l^j &= \frac{1}{2} (\delta_i^k P_{jl} - \delta_j^k P_{il} - 2P_{[ij]} \delta_l^k) \omega^i \wedge \omega^j, \end{aligned}$$

where – by slight abuse of notation – the $\mathbb{R}_2 \otimes \mathbb{R}_2$ -valued map $P = (P_{ij})$ represents the Schouten tensor of ∇ . We define a right G -action on $F \times \mathbb{R}_2$ by the rule

$$(u, \xi) \cdot (b \rtimes a) = (\det a^{-1} a^{-1} \circ u, \xi a \det a + b \det a),$$

for all $b \rtimes a \in G$. Denoting by $\pi : F \times \mathbb{R}_2 \rightarrow N$ the basepoint projection of the first factor, this action turns $\pi : F \times \mathbb{R}_2 \rightarrow N$ into a principal right G -bundle over N . On $F \times \mathbb{R}_2$ we define the $\mathfrak{sl}(3, \mathbb{R})$ -valued 1-form

$$\theta = \begin{pmatrix} -\frac{1}{3} \mathrm{tr} \varphi + \xi \omega & -d\xi + \xi \varphi - P^t \omega - \xi \omega \xi \\ \omega & \varphi - \frac{1}{3} \mathrm{I} \mathrm{tr} \varphi - \omega \xi \end{pmatrix}.$$

Then $(\pi : F \times \mathbb{R}_2 \rightarrow N, \theta)$ is a Cartan geometry of type $(\mathrm{SL}(3, \mathbb{R}), G)$ satisfying the properties (iii) to (v) for the projective structure defined by ∇ . It follows from the uniqueness part of Cartan's construction that $(\pi : F \times \mathbb{R}_2 \rightarrow N, \theta)$ is isomorphic to Cartan's bundle.

Note that we may think of the left action of $G \subset \mathrm{SL}(3, \mathbb{R})$ on \mathbb{R}_3 by matrix multiplication as a (linear) G -representation and consequently, we obtain an associated rank-3 vector bundle E for every projective surface $(N, [\nabla])$. The vector bundle E is commonly referred to as the *tractor bundle* of $(N, [\nabla])$. Interest in E stems from the fact that it comes canonically equipped with an $\mathrm{SL}(3, \mathbb{R})$ connection which is flat if and only if $(N, [\nabla])$ is. We refer the reader to [2] for additional details.

2.4. Anti-self-duality. Let M be an oriented four-dimensional manifold with a metric g of signature $(2, 2)$. The Hodge $*$ operator is an involution on two-forms, and induces a decomposition

$$\Lambda^2(T^*M) = \Lambda_+^2(T^*M) \oplus \Lambda_-^2(T^*M) \quad (2.6)$$

of two-forms into self-dual (SD) and anti-self-dual (ASD) components, which only depends on the conformal class of g . The Riemann tensor of g has the symmetry $R_{abcd} = R_{[ab][cd]}$ so can be thought of as a map $\mathcal{R} : \Lambda^2(T^*M) \rightarrow \Lambda^2(T^*M)$ which admits a decomposition under (2.6):

$$\mathcal{R} = \left(\begin{array}{c|c} C_+ - 2\Lambda & \phi \\ \hline \phi & C_- - 2\Lambda \end{array} \right). \quad (2.7)$$

Here C_\pm are the SD and ASD parts of the (conformal) Weyl tensor, ϕ is the trace-free Ricci curvature, and -24Λ is the scalar curvature which acts by scalar multiplication. The metric g is ASD if $C_+ = 0$. It is ASD and Einstein if $C_+ = 0$ and $\phi = 0$. Finally it is ASD Ricci-flat (or equivalently hyper-symplectic) if $C_+ = \phi = \Lambda = 0$. In this case the Riemann tensor is also anti-self-dual.

Locally there exist real rank-two vector bundles \mathbb{S}, \mathbb{S}' (spin-bundles) over M equipped with parallel symplectic structures $\varepsilon, \varepsilon'$ such that

$$TM \cong \mathbb{S} \otimes \mathbb{S}' \quad (2.8)$$

is a canonical bundle isomorphism, and

$$g(v_1 \otimes w_1, v_2 \otimes w_2) = \varepsilon(v_1, v_2)\varepsilon'(w_1, w_2)$$

for $v_1, v_2 \in \Gamma(\mathbb{S})$ and $w_1, w_2 \in \Gamma(\mathbb{S}')$. A vector $V \in \Gamma(TM)$ is called null if $g(V, V) = 0$. Any null vector is of the form $V = \lambda \otimes \pi$ where λ , and π are sections of \mathbb{S} and \mathbb{S}' respectively. An α -plane (respectively a β -plane) is a two-dimensional plane in $T_p M$ spanned by null vectors of the above form with π (respectively λ) fixed, and an α -surface (β -surface) is a two-dimensional surface in $\zeta \subset M$ such that its tangent plane at every point is an α -plane (β -plane). The seminal theorem of Penrose [31] states that a maximal, three dimensional, family of α -surfaces exists in M iff $C_+ = 0$.

3. FROM PROJECTIVE TO BI-LAGRANGIAN STRUCTURES

In this section we show how to canonically construct a bi-Lagrangian structure on the total space of a certain rank 2 affine bundle over a projective surface $(N, [\nabla])$.

For what follows it will be necessary to have an interpretation of vector fields on N in terms of G -equivariant maps on $B_{[\nabla]}$. To this end let

$$(b \rtimes a)^{-1} = \begin{pmatrix} \det a & -(\det a)ba^{-1} \\ 0 & a^{-1} \end{pmatrix}.$$

Hence

$$(R_{b \rtimes a})^* \theta = \begin{pmatrix} \det a & -(\det a)ba^{-1} \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} -\operatorname{tr} \phi & \eta \\ \omega & \phi \end{pmatrix} \begin{pmatrix} \det a^{-1} & b \\ 0 & a \end{pmatrix}$$

the right hand side of which becomes

$$\begin{pmatrix} -\det a \operatorname{tr} \phi - (\det a)ba^{-1}\omega & (\det a)\eta - (\det a)ba^{-1}\phi \\ a^{-1}\omega & a^{-1}\phi \end{pmatrix} \begin{pmatrix} \det a^{-1} & b \\ 0 & a \end{pmatrix}$$

and thus

$$(R_{b \rtimes a})^* \theta = \begin{pmatrix} -\operatorname{tr} \phi - b \cdot a^{-1}\omega & * \\ (\det a^{-1})a^{-1}\omega & a^{-1}\omega b + a^{-1}\phi a \end{pmatrix} \quad (3.1)$$

with

$$* = -(\det a)b \operatorname{tr} \phi - (\det a)ba^{-1}\omega b + \eta(\det a)a - (\det a)ba^{-1}\phi a. \quad (3.2)$$

Since

$$(R_{b \rtimes a})^* \omega = (\det a^{-1})a^{-1}\omega, \quad (3.3)$$

it follows that the tangent bundle of N is the bundle associated to the representation $\chi : G \rightarrow \operatorname{GL}(2, \mathbb{R})$, where χ is defined by the rule

$$\chi(b \rtimes a)x = (\det a)ax,$$

for all $x \in \mathbb{R}^2$ and $b \rtimes a \in G$. Consequently, the vector fields on N are represented by smooth \mathbb{R}^2 -valued functions X on $B_{[\nabla]}$ that are equivariant with respect to χ , that is,

$$(R_{b \rtimes a})^* X = \chi((b \rtimes a)^{-1})X = (\det a^{-1})a^{-1}X \quad (3.4)$$

for all $b \rtimes a \in G$.

Recall that the group G also acts faithfully on \mathbb{R}_2 by affine transformations defined by the rule

$$(b \rtimes a) \cdot \xi = \xi a^{-1} \det a^{-1} - ba^{-1}$$

for $b \rtimes a \in G$. Therefore, the bundle associated to $B_{[\nabla]}$ via this affine G -action is a rank-2 affine bundle $M \rightarrow N$ and from (3.4) we see that its associated vector bundle is the cotangent bundle of N . We will refer to M as the *canonical affine bundle* of $(N, [\nabla])$. Note that by construction $M \subset \mathbb{P}(E)$.

By definition, an element of M is an equivalence class $[u, \xi]$ with $u \in B_{[\nabla]}$ and $\xi \in \mathbb{R}_2$ subject to the equivalence relation

$$(u_1, \xi_1) \sim (u_2, \xi_2) \iff u_2 = u_1 \cdot b \rtimes a \quad \wedge \quad \xi_2 = (b \rtimes a)^{-1} \cdot \xi_1, \quad b \rtimes a \in G.$$

Clearly, every element of M has a representative $(u, 0)$, unique up to a $\operatorname{GL}(2, \mathbb{R})$ transformation, where here $\operatorname{GL}(2, \mathbb{R}) \subset G$ consists of those elements $b \rtimes a \in G$ satisfying $b = 0$. For simplicity of notation we will henceforth write a instead of $0 \rtimes a$ for the elements of

$\mathrm{GL}(2, \mathbb{R}) \subset G$. It follows that as a smooth manifold, M is canonically diffeomorphic to the quotient $B_{[\nabla]}/\mathrm{GL}(2, \mathbb{R})$.

Remark 3.1. It can be shown that the sections of $M \rightarrow N$ are in one-to-one correspondence with the $[\nabla]$ -representative connections. The submanifold geometry in M of representative connections will be studied in depth in a forthcoming article by the second author [29].

3.1. A bundle embedding. It turns out that we can embed $B_{[\nabla]} \rightarrow M$ as subbundle of the coframe bundle $F \rightarrow M$ of M . Here, we define a coframe at $p \in M$ to be a linear isomorphism $T_p M \rightarrow \mathbb{R}_2 \oplus \mathbb{R}^2$ and we denote the tautological $\mathbb{R}_2 \oplus \mathbb{R}^2$ -valued 1-form on F by ζ .

By definition of M , a vector field X on M is represented by a unique $(\mathbb{R}_2 \oplus \mathbb{R}^2)$ -valued function (X_+, X_-) on $B_{[\nabla]}$ satisfying the equivariance condition

$$R_a^* X_+ = X_+ a \det a, \quad R_a^* X_- = (\det a^{-1}) a^{-1} X_-. \quad (3.5)$$

Therefore, we obtain a unique map $\psi : B_{[\nabla]} \rightarrow F$ having the property that for every vector field X on M and for all $u \in B_{[\nabla]}$

$$\psi(u)(X(\mu(u))) = (X_+(u), X_-(u)),$$

where (X_+, X_-) is the function on $B_{[\nabla]}$ representing X . Clearly, ψ is a smooth embedding. Furthermore, from (3.5) we obtain

$$\psi(u \cdot a) = \psi(u) \cdot \chi(a)$$

where $\chi : G \ni \mathrm{GL}(2, \mathbb{R}) \rightarrow \mathrm{Aut}(\mathbb{R}_2 \oplus \mathbb{R}^2)$ is the Lie group embedding defined by the rule

$$\chi(a)(\xi, x) = (\xi a \det a, (\det a^{-1}) a^{-1} x).$$

Consequently, the pair (ψ, χ) embeds $B_{[\nabla]} \rightarrow M$ as a subbundle of the coframe bundle of M whose structure group is isomorphic to $\mathrm{GL}(2, \mathbb{R})$. Furthermore, unraveling the definition of ζ , it follows that we have

$$\psi^* \zeta = (\eta, \omega). \quad (3.6)$$

The induced geometric structure on M defined by the reduction of the coframe bundle of M is a bi-Lagrangian structure, so we will study these structures next.

3.2. Bi-Lagrangian structures. A *bi-Lagrangian* structure on smooth 4-manifold M (or more generally an even dimensional manifold) consists of a symplectic structure Ω together with a splitting of the tangent bundle of M into a direct sum of Ω -Lagrangian subbundles E_{\pm}

$$TM = E_+ \oplus E_-.$$

A manifold equipped with a bi-Lagrangian structure will be called a bi-Lagrangian manifold. The endomorphism $I : TM \rightarrow TM$ defined by

$$v = v_+ + v_- \mapsto v_+ - v_-, \quad v \in TM, v_{\pm} \in E_{\pm}$$

is Ω -skew-symmetric, satisfies $I^2 = \text{Id}$ and its ± 1 -eigenbundle is E_{\pm} . Clearly, I is the unique endomorphism of the tangent bundle having these properties and therefore, we may equivalently think of a bi-Lagrangian structure as a pair (Ω, I) consisting of a symplectic structure Ω and a Ω -skew-symmetric endomorphism $I : TM \rightarrow TM$ whose square is the identity.

Note also, that we may use the pair (Ω, I) to define a pseudo-Riemannian metric

$$g(v, w) = \Omega(v, I(w)), \quad v, w \in TM,$$

whose signature is $(2, 2)$ and for which I is skew-symmetric. Of course, a bi-Lagrangian structure is also equivalently described in terms of the pair (g, I) or the pair (g, Ω) .

Remark 3.2. What we call a bi-Lagrangian structure is also referred to as an *almost para-Kähler structure* and a *para-Kähler structure* provided E_{\pm} are both Frobenius integrable. Note that in [4] the term bi-Lagrangian structure is reserved for the case where both E_{\pm} are Frobenius integrable.

A bi-Lagrangian structure admits an interpretation as a reduction of the structure group of the coframe bundle of M . To this end consider the symmetric bilinear form of signature $(2, 2)$ on $\mathbb{R}_2 \oplus \mathbb{R}^2$

$$\langle (\xi_1, x^1), (\xi_2, x^2) \rangle = -\frac{1}{2} (\xi_1 x^2 + \xi_2 x^1)$$

and the skew-symmetric non-degenerate bilinear form

$$\rangle (\xi_1, x^1), (\xi_2, x^2) \langle = \frac{1}{2} (\xi_1 x^2 - \xi_2 x^1).$$

The two bilinear forms are related by the endomorphism ι sending $(\xi, x) \mapsto (\xi, -x)$. The endomorphism ι satisfies $\iota^2 = \text{Id}$ and its 1-eigenspace is $\mathbb{R}_2 \oplus \{0\}$ and its -1 -eigenspace is $\{0\} \oplus \mathbb{R}^2$. By construction, both eigenspaces are null and Lagrangian, that is, both bilinear forms vanish identically when restricted to the ι -eigenspaces. The group $\text{GL}(2, \mathbb{R})$ acts from the left on $\mathbb{R}_2 \oplus \mathbb{R}^2$ by

$$a \cdot (\xi, x) = (\xi a^{-1}, ax)$$

and this action preserves both bilinear forms. We henceforth identify $\text{GL}(2, \mathbb{R})$ with its image subgroup in $\text{Aut}(\mathbb{R}_2 \oplus \mathbb{R}^2)$. In fact, $\text{GL}(2, \mathbb{R})$ is the largest subgroup of $\text{Aut}(\mathbb{R}_2 \oplus \mathbb{R}^2)$ preserving both bilinear forms.

Given a bi-Lagrangian structure (Ω, I) on M we say that a coframe u at $p \in M$ is *adapted* to (Ω, I) if for all $v, w \in T_p M$

$$\Omega_p(v, w) = \rangle u(v), u(w) \langle \quad \text{and} \quad (u \circ I)(v) = (\iota \circ u)(v).$$

The set of all coframes of M adapted to (Ω, I) defines a reduction $\lambda : B_{(\Omega, I)} \rightarrow M$ of the coframe bundle $F \rightarrow M$ of M with structure group $\text{GL}(2, \mathbb{R})$. Conversely, every reduction of the coframe bundle of M with structure group $\text{GL}(2, \mathbb{R})$ defines a unique pair (Ω, I) ,

consisting of a non-degenerate 2-form on M and a Ω -skew symmetric endomorphism $I : TM \rightarrow TM$ whose square is the identity. Note however that Ω need not be closed.

The tautological $\mathbb{R}_2 \oplus \mathbb{R}^2$ -valued 1-form ζ on $B_{(\Omega, I)}$ will be written as $\zeta = (\eta, \omega)$, so that $\eta = (\eta_i)$ is an \mathbb{R}_2 -valued 1-form on $B_{(\Omega, I)}$ and $\omega = (\omega^i)$ is an \mathbb{R}^2 -valued 1-form on $B_{(\Omega, I)}$. By construction, we have

$$\lambda^* \Omega = -\eta \wedge \omega \equiv -\eta_i \wedge \omega^i.$$

Furthermore, let $\hat{L}_\pm = (\lambda')^{-1}(E_\pm) \subset TB_{(\Omega, I)}$, then the subbundle E_+ is defined by the equations $\eta = 0$ and the subbundle E_- is defined by the equations $\omega = 0$.

A linear connection on F is said to be adapted to (Ω, I) if it pulls back to $B_{(\Omega, I)}$ to become a principal $\mathrm{GL}(2, \mathbb{R})$ -connection on $B_{(\Omega, I)}$. An adapted connection is given by a $\mathfrak{gl}(2, \mathbb{R})$ -valued equivariant 1-form ν on $B_{(\Omega, I)}$ such that

$$\begin{aligned} d\eta &= -\eta \wedge \nu + \frac{1}{2}T_+((\eta, \omega) \wedge (\eta, \omega)), \\ d\omega &= -\nu \wedge \omega + \frac{1}{2}T_-((\eta, \omega) \wedge (\eta, \omega)), \end{aligned}$$

for some torsion map T_+ on $B_{(\Omega, I)}$ with values in $\mathrm{Hom}(\Lambda^2(\mathbb{R}_2 \oplus \mathbb{R}^2), \mathbb{R}_2)$ and some torsion map T_- on $B_{(\Omega, I)}$ with values in $\mathrm{Hom}(\Lambda^2(\mathbb{R}_2 \oplus \mathbb{R}^2), \mathbb{R}^2)$, both of which are equivariant with respect to the $\mathrm{GL}(2, \mathbb{R})$ right action. It is an easy exercise in linear algebra to check that for every bi-Lagrangian structure there exists a unique adapted connection ν so that

$$\begin{aligned} d\eta &= -\eta \wedge \nu + \frac{1}{2}T_+(\omega \wedge \omega), \\ d\omega &= -\nu \wedge \omega + \frac{1}{2}T_-(\eta \wedge \eta), \end{aligned} \tag{3.7}$$

with T_+ taking values in $\mathrm{Hom}(\Lambda^2 \mathbb{R}^2, \mathbb{R}_2)$ and T_- taking values in $\mathrm{Hom}(\Lambda^2 \mathbb{R}_2, \mathbb{R}^2)$. It follows that E_\pm is integrable if and only if T_\pm vanishes identically. Furthermore, the identity $d(\eta \wedge \omega) = 0$ implies

$$T_+(\omega \wedge \omega) \wedge \omega = 0 \quad \text{and} \quad T_-(\eta \wedge \eta) \wedge \eta = 0.$$

The linear connection ν on the bundle of adapted frames induces connections on the tensor bundles of M in the usual way. By construction, the induced connection ${}^\nu \nabla$ on TM is the unique (affine) connection with torsion τ satisfying

$${}^\nu \nabla \Omega = 0 \quad \text{and} \quad {}^\nu \nabla I = 0 \quad \text{and} \quad \tau(X_+, X_-) = 0,$$

for all $X_\pm \in \Gamma(E_\pm)$. To the best of our knowledge, the connection ${}^\nu \nabla$ was first studied by Libermann [26], so we call ν the *Libermann connection*. Of course, if τ vanishes identically, then ${}^\nu \nabla$ is just the Levi-Civita connection of g .

3.3. From projective to bi-Lagrangian structures. Denoting by $B_{(\Omega, I)}$ the bundle of adapted coframes of a bi-Lagrangian structure (Ω, I) and by $B_{[\nabla]}$ the Cartan bundle of a projective structure $[\nabla]$, we obtain:

Theorem 3.3. *Let $(N, [\nabla])$ be a projective surface with Cartan bundle $(\pi : B_{[\nabla]} \rightarrow N, \theta)$. Then there exists a bi-Lagrangian structure (Ω, I) on the quotient $M = B_{[\nabla]}/\mathrm{GL}(2, \mathbb{R})$ having the following property: There exists a $\mathrm{GL}(2, \mathbb{R})$ -bundle isomorphism $\psi : B_{[\nabla]} \rightarrow B_{(\Omega, I)}$ so that*

$$\psi^* \begin{pmatrix} -\frac{1}{3} \mathrm{tr} \nu & \eta \\ \omega & \nu - \frac{1}{3} \mathrm{Id} \mathrm{tr} \nu \end{pmatrix} = \theta,$$

where (η, ω) denotes the tautological 1-form on $B_{(\Omega, I)}$ and ν the Libermann connection. Moreover, the E_- -bundle of the bi-Lagrangian structure (Ω, I) is always integrable and the E_+ -bundle is integrable if and only if $[\nabla]$ is flat.

Proof. We write

$$\theta = \begin{pmatrix} -\mathrm{tr} \phi & \hat{\eta} \\ \hat{\omega} & \phi \end{pmatrix}$$

for the Cartan connection on $B_{[\nabla]}$. From §3.1 we know that we have an embedding (ψ, χ) of $B_{[\nabla]} \rightarrow M$ as a $\mathrm{GL}(2, \mathbb{R})$ -subbundle $\lambda : B_{(\Omega, I)} \rightarrow M$ of the coframe bundle of M . Furthermore, ψ satisfies

$$\psi^*(\eta, \omega) = (\hat{\eta}, \hat{\omega}),$$

where (η, ω) denotes the tautological $(\mathbb{R}_2 \oplus \mathbb{R}^2)$ -valued 1-form on $B_{(\Omega, I)}$. Therefore, we obtain a unique non-degenerate 2-form Ω on M and a unique Ω -skew-symmetric endomorphism $I : TM \rightarrow TM$ whose square is the identity. The 2-form Ω pulled back to $B_{(\Omega, I)}$ becomes $-\eta \wedge \omega$. The structure equations (2.4) imply that we have

$$\begin{aligned} d\hat{\omega} &= -(\phi + I \mathrm{tr} \phi) \wedge \hat{\omega}, \\ d\hat{\eta} &= -\hat{\eta} \wedge (\phi + I \mathrm{tr} \phi) + L(\hat{\omega} \wedge \hat{\omega}). \end{aligned} \tag{3.8}$$

In particular, we obtain

$$\begin{aligned} d(\hat{\eta} \wedge \hat{\omega}) &= [-\hat{\eta} \wedge (\phi + I \mathrm{tr} \phi) + L(\hat{\omega} \wedge \hat{\omega})] \wedge \hat{\omega} - \hat{\eta} \wedge [-(\phi + I \mathrm{tr} \phi) \wedge \hat{\omega}] \\ &= L(\hat{\omega} \wedge \hat{\omega}) \wedge \hat{\omega} = 0, \end{aligned}$$

where the last equality follows since N is two-dimensional. This shows that Ω is symplectic, so that the pair (Ω, I) defines a bi-Lagrangian structure on M . The equivariance properties of θ and (3.8) imply that the ψ -pushforward of $\phi + I \mathrm{tr} \phi$ is a principal right $\mathrm{GL}(2, \mathbb{R})$ -connection on $B_{(\Omega, I)}$ which satisfies (3.7) with $T_- \equiv 0$ and $T_+ = L \circ \psi^{-1}$. In particular, E_- is always integrable and E_+ is integrable if and only if L vanishes identically, that is, $[\nabla]$ is flat. Denoting by ν the Libermann connection of (Ω, I) , we obtain from its uniqueness that

$$\psi^* \nu = \phi + I \mathrm{tr} \phi, \tag{3.9}$$

which completes the proof. \square

Denoting by $\rho : M \rightarrow N$ the basepoint projection, we immediately obtain:

Corollary 3.4. *For every open set $U \subset N$ the Lie algebra of projective vector fields $\mathcal{P}_{[\nabla]}(U)$ is isomorphic to the Lie algebra of bi-Lagrangian vector fields $\mathcal{B}_{(\Omega, I)}(\rho^{-1}(U))$.*

Remark 3.5. Here we call a vector field X defined on some open subset $U \subset (N, [\nabla])$ *projective*, if its flow maps unparametrised geodesics of $[\nabla]$ to unparametrised geodesics of $[\nabla]$. The set of such vector fields on U is a Lie subalgebra of the Lie algebra of vector fields on U which we will denote by $\mathcal{P}_{[\nabla]}(U)$. Likewise, we call a vector field X defined on some open subset $V \subset (M, \Omega, I)$ *bi-Lagrangian* if its (local) flow preserves both Ω and I . The set of such vector fields on V is a Lie subalgebra of the Lie algebra of vector fields on V which we will denote by $\mathcal{B}_{(\Omega, I)}(V)$.

Proof of Corollary 3.4. By standard results about Cartan geometries (c.f. [9]), the projective vector fields on $U \subset (N, [\nabla])$ are in one-to-one correspondence with the vector fields on $\pi^{-1}(U) \subset B_{[\nabla]}$ whose flow preserves the Cartan connection θ . Theorem 3.3 implies that such a vector field corresponds to a vector field on $\psi(\pi^{-1}(U)) \subset B_{(\Omega, I)}$ preserving both the tautological form (η, ω) and the Libermann connection. Again, by standard results about G -structures [9], such vector fields are in one-to-one correspondence with vector fields on $\rho^{-1}(U)$ preserving both Ω and I . \square

Remark 3.6. In addition to the proof of Corollary 3.4 relying on standard results about G -structures and Cartan geometries, we will provide an explicit lift of a projective vector field on N to a bi-Lagrangian vector field on M in Theorem 6.3. In addition, in §6 a stronger result will be established: any Killing vector on (M, g) is necessarily bi-Lagrangian, and arises from some projective vector field on N .

Remark 3.7. Note that if X_x is a vector field on $B_{[\nabla]}$ having the property that

$$\omega(X_x) = x, \quad \eta(X_x) = 0, \quad \phi(X_x) = 0,$$

for some non-zero $x \in \mathbb{R}^2$, then the integral curve of X_x , when projected to N , becomes a geodesic of $[\nabla]$. Conversely every geodesic of $[\nabla]$ arises in this way (see for instance [23] for details). Likewise, a geodesic of the Libermann connection arises as the projection of an integral curve of a horizontal vector field on $B_{(\Omega, I)}$ which is constant on the canonical 1-form. It follows that the geodesics on $(N, [\nabla])$ correspond to the geodesics of the Libermann connection on (M, Ω, I) that are everywhere tangent to E_- .

3.4. The choice of a representative connection. Recall from §2.3 that the choice of a representative connection $\nabla \in [\nabla]$ gives a G -bundle isomorphism $B_{[\nabla]} \simeq F \times \mathbb{R}_2$. In particular, we obtain an affine bundle isomorphism $\psi : (F \times \mathbb{R}_2)/\text{GL}(2, \mathbb{R}) \rightarrow M$. By construction, the quotient $(F \times \mathbb{R}_2)/\text{GL}(2, \mathbb{R})$ is just the cotangent bundle of N . Denoting by $\lambda : F \times \mathbb{R}_2 \rightarrow T^*N$ the basepoint projection, we obtain

$$\begin{aligned} (\psi \circ \lambda)^* g &= -(-d\xi + \xi\varphi - P^t\omega - \xi\omega\xi) \odot \omega, \\ (\psi \circ \lambda)^* \Omega &= \omega \wedge (-d\xi + \xi\varphi - P^t\omega - \xi\omega\xi), \end{aligned} \tag{3.10}$$

where the $\mathbb{R}_2 \otimes \mathbb{R}_2$ -valued map $P = (P_{ij})$ on F represents the Schouten tensor and the φ the connection form of ∇ . A choice of local coordinates $x = (x^i) : U \rightarrow \mathbb{R}^2$ on N induces a trivialisation $v^{-1}(U) \simeq U \times \mathrm{GL}(2, \mathbb{R})$ so that

$$\omega = a^{-1}dx, \quad \varphi = a^{-1}da + a^{-1}\Gamma dx a,$$

where $\Gamma = (\Gamma_{jk}^i)$ denote the Christoffel symbols of ∇ with respect to x and $a : U \times \mathrm{GL}(2, \mathbb{R}) \rightarrow \mathrm{GL}(2, \mathbb{R})$ the projection onto the latter factor. Taking the quotient by $\mathrm{GL}(2, \mathbb{R})$, we obtain from (3.10) the following coordinate expressions

$$\begin{aligned} g &= (d\xi_i \odot dx^i - (\xi_l \Gamma_{ij}^l - P_{(ij)} - \xi_i \xi_j) dx^i \odot dx^j), \\ \Omega &= d\xi_i \wedge dx^i + P_{[ij]} dx^i \wedge dx^j. \end{aligned} \tag{3.11}$$

4. GAUGE THEORY ON SURFACES WITH PROJECTIVE STRUCTURE

Let $(N, [\nabla])$ be a projective structure on a surface, and let $E \rightarrow N$ be a vector bundle with \mathfrak{g} -connection A , where \mathfrak{g} is some Lie algebra. Let ϕ be a one-form on N , called the Higgs pair, with values in the adjoint bundle $\mathrm{adj}(E)$. In an open set $U \subset N$ we shall write $\phi = \phi_i dx^i$. Let $\nabla \in [\nabla]$ be a chosen connection in the projective class. The Calderbank equations [7, 8] hold if

$$D_{(i}\phi_{j)} = 0, \tag{4.1}$$

where

$$D_i \phi_j := \partial_i \phi_j - \Gamma_{ij}^k \phi_k - [A_i, \phi_j].$$

Equivalently, the Higgs pair is constant along the charged geodesic spray on TN i.e.

$$\Theta^A(\phi) := \left(\pi^i \frac{\partial}{\partial x^i} - \Gamma_{ij}^k \pi^i \pi^j \frac{\partial}{\partial \pi^k} \right) (\phi) - [A, \phi] = 0 \tag{4.2}$$

where π^i are coordinates on the fibres of TN , and $\phi = \phi_i \pi^i$ and $A = A_i \pi^i$ are \mathfrak{g} -valued linear functions on TN . The equations (4.1) do not depend on the choice of the connection ∇ in the projective class if the Higgs pair ϕ has projective weight 2.

In the §5 we shall show how the Calderbank equations with the gauge group $\mathrm{SL}(3, \mathbb{R})$ regarded as a subgroup of the group of diffeomorphisms of \mathbb{RP}^2 leads to neutral signature anti-self-dual Einstein metrics in four dimensions, which are also bi-Lagrangian and given by (3.11). We shall first list some other (implicit) occurrences of these equations for other gauge groups.

4.1. Null reductions of anti-self-dual Yang–Mills equations. If the projective structure is flat, then (4.1) is the symmetry reduction of the anti-self-dual Yang–Mills (ASDYM) equation on $\mathbb{R}^{2,2}$ by two null translations and such that the $(2, 2)$ metric g restricted to the two-dimensional space of orbits $N = \mathbb{R}^2$ is totally isotropic, and the bi-vector generated by the null translations is anti-self-dual.

To see it, consider a \mathfrak{g} -valued connection one-form A on $\mathbb{R}^{2,2}$, and set $F = dA + A \wedge A$. In local coordinates adapted to $\mathbb{R}^{2,2} = TN$ with x^i the coordinates on N , the null isometries are $\partial/\partial\xi_i$, and the metric is

$$g = dx^1 d\xi_1 + dx^2 d\xi_2.$$

Choose an orientation on $\mathbb{R}^{2,2}$ such that the two-form $dx^1 \wedge dx^2$ is ASD. Defining two Higgs fields $\phi_1 = \partial/\partial\xi_2 \lrcorner A$, $\phi_2 = \partial/\partial\xi_1 \lrcorner A$, the ASDYM equations $F = - * F$ yield [28]

$$D_1\phi_1 = 0, \quad D_2\phi_2 = 0, \quad D_1\phi_2 + D_2\phi_1 = 0, \quad (4.3)$$

where $D = d + A_1 dx^1 + A_2 dx^2$ is a covariant derivative on N induced by A . In [34] these equations have been solved completely for the gauge group $SL(2)$.

4.1.1. Prolongation of the Calderbank equations. Instead of regarding both the connection and the Higgs pair as unknowns we shall assume that the connection is given and consider (4.3) as a system of PDEs for the Higgs pair. To determine all derivatives of the Higgs pair in (4.3) we prolong the system once, and define a section μ of $\text{adj}(E)$ by

$$D_i\phi_j = \frac{1}{2}\mu\epsilon_{ij},$$

where $\epsilon = dx^1 \wedge dx^2$ is the parallel volume form of $\nabla \in [\nabla]$. Commuting the covariant derivatives now leads to a closed system and therefore a connection

$$D_i \begin{pmatrix} \phi_j \\ \mu \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\mu\epsilon_{ij} \\ 2[\phi_i, \mathcal{F}] \end{pmatrix},$$

where $\mathcal{F} = [D_1, D_2]$ is the \mathfrak{g} -valued curvature of the connection A . The system is now closed. Commuting the covariant derivatives on μ leads to an integrability condition

$$[\mathcal{F}, \mu] - 2D_1[\phi_2, \mathcal{F}] + 2D_2[\phi_1, \mathcal{F}] = 0.$$

4.2. Killing equations. If the connection A is flat, and $\mathfrak{g} = \mathbb{R}$ then the Calderbank equations become the projectively invariant Killing equations.

4.3. Anti-self-dual conformal structures with null conformal Killing vectors. Let \mathfrak{g} be a subalgebra of infinite dimensional Lie algebra of vector fields $\mathfrak{diff}(\Sigma)$ of vector fields on a surface Σ consisting of those elements of $\mathfrak{diff}(\Sigma)$ which commute with a fixed vector field K on Σ . Let $M \rightarrow N$ be a surface bundle over N , with two dimensional fibres Σ . In this case the Calderbank equations are solvable by quadrature and the two-dimensional distribution

$$\mathcal{D} = \{\Theta^A := \pi^i \frac{\partial}{\partial x^i} - \Gamma_{ij}^k \pi^i \pi^j \frac{\partial}{\partial \pi^k} - A_i(x) \pi^i, \phi = \pi^i \phi_i\} \quad (4.4)$$

spanning an \mathbb{RP}^1 worth of null self-dual surfaces (α -surfaces) through each point of M is the twistor distribution for the most general ASD $(2, 2)$ conformal structure which admits a null conformal Killing vector K [17, 7, 30].

4.4. Walker's Riemannian extensions. The conformal structure resulting from the distribution (4.4) is a generalisation of the Walker lift [38, 3]. To recover the Walker lift

$$g = d\xi_i \odot dx^i - \Gamma_{ij}^k \xi_k dx^i \odot dx^j, \quad (4.5)$$

take the gauge algebra $\mathfrak{g} = \mathfrak{gl}(2, \mathbb{R})$ which generates linear transformations of $\Sigma = \mathbb{R}^2$. If the coordinates on Σ are (ξ_1, ξ_2) , the elements of $\mathfrak{gl}(2, \mathbb{R})$ are vector fields of the form $\mathbf{t}_i^j = \xi_i \frac{\partial}{\partial \xi_j}$. Taking the connection A and the Higgs one-form given by

$$A_i = -\Gamma_{ij}^k \xi_k \frac{\partial}{\partial \xi_j}, \quad \phi_i = (b^k \xi_k) \epsilon_{ij} \frac{\partial}{\partial \xi_j}$$

where b^k is a non-zero constant leads to an integrable distribution (4.4), as then

$$\begin{aligned} [\Theta^A, \phi] &= \pi^i b^j \left(\Gamma_{jk}^i \xi_i + \Gamma_{ij}^k \xi_k \right) \epsilon_{lm} \pi^l \frac{\partial}{\partial \xi_m} \\ &= 0 \pmod{\phi}. \end{aligned}$$

The resulting metric (4.5) on $M = TN$ is then uniquely determined by the condition that the integral two-surfaces of \mathcal{D} in $TN \times \mathbb{RP}^1$ project down to self-dual totally null surfaces on TN . The generators of the gauge group satisfy $[\mathbf{v}, K] = \mathbf{v}$, where $K = \xi_1 \partial / \partial \xi_1 + \xi_2 \partial / \partial \xi_2$ is a conformal null Killing vector of (4.5).

We shall end this subsection by clarifying the connection between changing the projective class of $(N, [\nabla])$, and conformal rescalings of the metric g on T^*N . We shall restrict our discussion to *special connections* in $[\nabla]$ which preserve some volume. Consider the effect of transformation (2.1) with $\Upsilon_i = \nabla_i f$, together with rescaling the fibers of $TN \rightarrow N$

$$\xi_i \rightarrow \hat{\xi}_i = e^{2f} \xi_i$$

on the Walker lift² (4.5). A straightforward calculation yields

$$\hat{g} = e^{2f} g.$$

Thus conformal scales on TN correspond to projective scales on N .

²In [18] (see also [6, 20, 21] for other applications of this lift) it was proven that a ‘similar’ metric

$$g = d\xi_i \odot dx^i - \Pi_{ij}^k \xi_k dx^i \odot dx^j, \quad (4.6)$$

constructed out of the Thomas symbols $\Pi_{ij}^k = \Gamma_{ij}^k - \frac{1}{3} \Gamma_{il}^l \delta_j^k - \frac{1}{3} \Gamma_{jl}^l \delta_i^k$ is anti-self-dual and null-Kähler (with ASD null-Kähler two-form) for any choice of Γ_{ij}^k . The Walker lift (4.5) is conformally equivalent (up to a diffeomorphism) to the projective Walker lift (4.6) only if $\Gamma_{ij}^j = \nabla_i F$ for some function F on N .

5. TRACTOR CONNECTION AND ASD EINSTEIN METRICS

In this Section we shall consider the Calderbank equations, where the gauge group is $\mathrm{SL}(3, \mathbb{R})$, and E is the tractor bundle for the projective structure $[\nabla]$.

Let $\mathcal{E}(1)$ be the line bundle of projective densities of weight 1 (see Section 2.2). Consider a rank-three vector bundle $E = \mathcal{E}(1) \oplus (T^*N \otimes \mathcal{E}(1))$ over N with connection [2]

$$\mathcal{D}_i \begin{pmatrix} \sigma \\ \mu_j \end{pmatrix} = \begin{pmatrix} \nabla_i \sigma - \mu_i \\ \nabla_i \mu_j + P_{ij} \sigma \end{pmatrix}, \quad (5.1)$$

where P_{ij} is the (not necessarily symmetric) Schouten tensor of projective geometry. The splitting of the tractor bundle depends on a choice of a connection ∇ in the projective class $[\nabla]$, and under (2.1) changes according to

$$\begin{pmatrix} \hat{\sigma} \\ \hat{\mu}_j \end{pmatrix} = \begin{pmatrix} \sigma \\ \mu_j + \Upsilon_j \sigma \end{pmatrix}. \quad (5.2)$$

Using the tractor indices $\alpha, \beta, \dots = 0, 1, 2$ we can rewrite the connection (5.1) in terms of its Christoffel symbols $\gamma_{i\alpha}^\beta$ as

$$\gamma_{i0}^0 = 0, \quad \gamma_{i0}^j = \delta_i^j, \quad \gamma_{ij}^k = \Gamma_{ij}^k, \quad \gamma_{ij}^0 = -P_{ij}.$$

The vector fields

$$\mathbf{t}_\alpha^\beta = \psi_\alpha \frac{\partial}{\partial \psi_\beta}$$

generate the linear action of $\mathrm{GL}(3, \mathbb{R})$ on the fibres of E . These generators descent to eight vector fields (which we shall also denote \mathbf{t}_α^β) which generate the action of $\mathrm{SL}(3, \mathbb{R})$ on the fibres of the projective tractor bundle $\mathbb{P}(E)$ which is a quotient of E by the Euler vector field $\sum_{\alpha=0}^2 \mathbf{t}_\alpha^\alpha$. Setting $\xi_i = \psi_i / \psi_0$ yields

$$\mathbf{t}_i^j = \xi_i \frac{\partial}{\partial \xi_j}, \quad \mathbf{t}_i^0 = -\xi_i \xi_j \frac{\partial}{\partial \xi_j}, \quad \mathbf{t}_0^i = \frac{\partial}{\partial \xi_i}, \quad \mathbf{t}_0^0 = -\xi_j \frac{\partial}{\partial \xi_j}.$$

Consider the Calderbank equations with the gauge group $\mathrm{SL}(3, \mathbb{R}) \subset \mathrm{Diff}(\mathbb{RP}^2)$, where the connection is given by a vector-valued one-form

$$A = A_i dx^i = -\gamma_{i\beta}^\alpha dx^i \otimes \mathbf{t}_\alpha^\beta$$

so that

$$A_i = (P_{ij} + \xi_i \xi_j - \Gamma_{ij}^k \xi_k) \frac{\partial}{\partial \xi_j}.$$

The Calderbank equations are solved by the Higgs pair

$$\phi_i = \epsilon_{ij} \frac{\partial}{\partial \xi_j}.$$

Let M be a complement of a projective line in the total space of the bundle $\mathbb{P}(E)$. The corresponding contravariant metric on M is constructed by demanding that the leaves of

the rank-2 distribution (4.4) $\mathcal{D} \subset T(M \times \mathbb{RP}^1)$ project down to self-dual two-surfaces on M . This gives $\epsilon^{ij}(\partial/\partial x^i - A_i) \odot \phi_j$, or, in the covariant form,

$$g = (d\xi_i - (\Gamma_{ij}^k \xi_k - \xi_i \xi_j - P_{ji}) dx^j) \odot dx^i, \quad (5.3)$$

so that we have recovered the bi-Lagrangian structure (3.11).

Theorem 5.1. *Formula (5.3) defines a metric which does not depend on a choice of a connection in a projective class.*

Proof. If we change the connection in the projective class using (2.1) then the Schouten tensor changes by (2.3). To establish the invariance of (5.3) we translate the fibre coordinates according to

$$\hat{\xi}_i = \xi_i + \Upsilon_i$$

in agreement with (5.2). Then

$$\begin{aligned} & (d\hat{\xi}_i - (\hat{\Gamma}_{ij}^k \hat{\xi}_k - \hat{\xi}_i \hat{\xi}_j - \hat{P}_{ji}) dx^j) \odot dx^i = d\xi_i \odot dx^i + \\ & \left(\xi_{(j} \Upsilon_{i)} - \Gamma_{ij}^k \xi_k - \xi_i \Upsilon_j - \xi_j \Upsilon_i - \Gamma_{ij}^k \Upsilon_k - 2\Upsilon_i \Upsilon_j + \xi_i \xi_j + \xi_i \Upsilon_j + \xi_j \Upsilon_i + \Upsilon_i \Upsilon_j \right. \\ & \left. + P_{ji} - \nabla_{(j} \Upsilon_{i)} + \Upsilon_i \Upsilon_j \right) dx^i \odot dx^j \\ & = (d\xi_i - (\Gamma_{ij}^k \xi_k - \xi_i \xi_j - P_{ji}) dx^j) \odot dx^i. \end{aligned}$$

□

The metric is anti-self-dual, and Einstein with scalar curvature equal to -24 . The anti-self-duality is a consequence of the fact that the connection A and the Higgs field $\phi_i \pi^i$ satisfy the Calderbank equations [7]. We shall establish the anti-self-duality for a larger class of metrics of the modified Walker form [11, 15], by verifying the integrability of the twistor distribution (4.4).

Proposition 5.2. *The conformal class defined by the metric*

$$g = (d\xi_i - (\Gamma_{ij}^k \xi_k - \xi_i \xi_j - M_{ij}) dx^j) \odot dx^i \quad (5.4)$$

is anti-self-dual for any $\Gamma_{ij}^k = \Gamma_{ij}^k(x^l)$ and $M_{ij} = M_{ij}(x^l)$. Moreover the metric (5.4) is Einstein with non-zero scalar curvature if and only if $M_{ij} = P_{(ij)}$ is the symmetrised projective Schouten tensor of Γ_{ij}^k .

Proof. We shall compute the Lie bracket

$$\begin{aligned} [\Theta^A, \phi] &= \left[\pi^i \left(\frac{\partial}{\partial x^i} + \Gamma_{ij}^k \xi_k \frac{\partial}{\partial \xi_j} - \xi_i \xi_j \frac{\partial}{\partial \xi_j} - M_{ij} \frac{\partial}{\partial \xi_j} \right) - \Gamma_{ij}^k \pi^i \pi^j \frac{\partial}{\partial \pi^k}, \pi^i \epsilon_{ij} \frac{\partial}{\partial \xi_j} \right] \\ &= -\pi^i \Gamma_{ij}^k \epsilon_{km} \left(\pi^j \frac{\partial}{\partial \xi_m} - \pi^m \frac{\partial}{\partial \xi_j} \right) + \xi_i \pi^i \pi^j \epsilon_{jk} \frac{\partial}{\partial \xi_k} \\ &= (-\pi^j \Gamma_{jk}^i + \pi^j \xi_j) \phi \in \text{span}\{\Theta^A, \phi\}. \end{aligned}$$

Therefore \mathcal{D} is Frobenius integrable, and there exists a one-parameter family of self-dual two-surfaces through each point of M , and (by Penrose's Nonlinear Graviton Theorem [31]) the conformal structure of g is ASD. The last part of the proposition is established by directly computing the Einstein tensor. \square

5.1. The characterisation of the projective-to-Einstein lift. A general, real analytic ASD Einstein metric in four dimensions depends on two arbitrary functions of three variables. If the projective structure on N is real analytic, then metrics of the form (1.1) depend on two arbitrary functions of two variables: two out of six functions Γ_{ij}^k can be eliminated by a diffeomorphism of N , and two more functions can be eliminated by making a projective change of connection (2.1) which, by Theorem 5.1, leaves the metric invariant. In this Section we shall invariantly characterise the ASD Einstein metrics arising from projective structures by the ansatz (1.1).

Recall [37, 3, 13] that a distribution $\mathcal{D} \subset TM$ is called parallel if ${}^g\nabla_X Y \in \Gamma(\mathcal{D})$ if $X \in \Gamma(\mathcal{D})$, where ${}^g\nabla$ is the Levi-Civita connection of g . Thus, if \mathcal{D} is parallel, then it is necessarily Frobenius integrable as $[X, Y] = {}^g\nabla_X Y - {}^g\nabla_Y X \in \Gamma(\mathcal{D})$ if $X, Y \in \Gamma(\mathcal{D})$.

Theorem 5.3. *Let (M, g) be an Einstein metric with scalar curvature scaled to -24 , such that Weyl tensor is ASD, and there exists a parallel ASD totally null distribution. Then g is conformally flat, or it is locally of the form (5.3), with $M = T^*N$ for some projective structure $[\nabla]$ on N .*

Proof. We shall rely on two isomorphisms: $TM = \mathbb{S} \otimes \mathbb{S}'$, and $\Lambda^2_- = \mathbb{S} \odot \mathbb{S}$. Let the ASD totally null distribution correspond to an ASD two-form $\Theta \in \Gamma(\Lambda^2_-)$, or equivalently to a section $\iota \in \Gamma(\mathbb{S})$. The Frobenius integrability conditions imply the local existence of two functions ξ_1 and ξ_2 on M such that $\text{Ker}(\Theta) = \text{span}\{\partial/\partial\xi_1, \partial/\partial\xi_2\}$. We can rescale ι so that the corresponding two-form is closed, and proportional to $dx^1 \wedge dx^2$ for some functions (x^1, x^2) which are constant on each β -surface in the two parameter family. The functions (ξ_1, ξ_2) are then the coordinates on the β -surface. The corresponding metric takes the form

$$g = d\xi_i \odot dx^i + \Theta_{ij}(x, \xi) dx^i \odot dx^j$$

for some symmetric two-by-two matrix Θ . The anti-self-duality condition on the Weyl tensor forces the components of Θ to be at most cubic in (ξ_1, ξ_2) , with some additional algebraic relations between the components. Imposing the Einstein condition (with non-zero Ricci scalar scaled to -24) gives $\Theta_{ij} = \xi_i \xi_j + P_{ji} - \Gamma_{ij}^k \xi_k$, where the functions Γ_{ij}^k do not depend on ps and are otherwise arbitrary, and the functions P_{ij} are determined by (2.2). \square

Remark 5.4. Existence of a neutral metric with a two-plane distribution impose topological restrictions on M . If M is compact then [1, 22]

$$\chi[M] \equiv 0 \pmod{2}, \quad \chi[M] \equiv \tau[M] \pmod{4},$$

where $\tau[M]$ and $\chi[M]$ are the signature and Euler characteristic respectively.

Remark 5.5. Rescaling the fibre coordinates in (5.3) allows us to change the Ricci scalar from -24 to -24Λ , where Λ is an arbitrary non-zero constant. This gives the metric (1.1)

$$g = \left(d\xi_i - \left(\Gamma_{ij}^k \xi_k - \Lambda \xi_i \xi_j - \frac{1}{\Lambda} P_{ji} \right) dx^j \right) \odot dx^i.$$

The one-parameter family of metrics (5.3) does not admit the Ricci flat limit unless $P_{(ij)} = 0$. The special case $P_{ij} = 0$ provides the homogeneous model of our construction which will be discussed in §7.1.

Remark 5.6. Performing the analogous rescaling on the bi-Lagrangian structure (3.11) gives the *charged* symplectic structure³ Ω given by (1.2). This two-form is ASD with respect to our choice of orientation, and the metric (1.1). We find that

$${}^g\nabla\Omega = 4L$$

where $L = \epsilon^{bc} \nabla_b P_{ca} dx^a \otimes (dx^1 \wedge dx^2)$ is the Liouville tensor which vanishes iff $[\nabla]$ is projectively flat.

6. KILLING VECTOR FIELDS FROM PROJECTIVE VECTOR FIELDS

In this Section we shall establish a correspondence between symmetries of a projective structure and isometries of the corresponding Einstein metric. As a spin-off from our analysis we shall show how to reduce a problem of solving a system of second order PDEs for projective symmetries, to a problem involving a system of first order PDEs for Killing vectors.

We shall first consider a correspondence between symmetries of affine connection and Killing vectors for a Walker metric (4.5). Let Γ_{ij}^k be an affine connection on N . A vector field K on N is called affine if

$$\mathcal{L}_K \Gamma_{ij}^k \equiv \frac{\partial^2 K^k}{\partial x^i \partial x^j} + K^m \frac{\partial}{\partial x^m} \Gamma_{ij}^k - \Gamma_{ij}^m \frac{\partial K^k}{\partial x^m} + \Gamma_{im}^k \frac{\partial K^m}{\partial x^j} + \Gamma_{jm}^k \frac{\partial K^m}{\partial x^i} = 0 \quad (6.1)$$

or equivalently if

$$\nabla_i \nabla_j K^k = R_{ij}{}^k{}_l K^l.$$

Any vector field on N corresponds to a linear function on T^*N , which in local coordinates is given by $K^i \xi_i$. This function, together with the canonical symplectic structure on T^*N gives rise to the Hamiltonian vector field

$$\tilde{K} = K^i \frac{\partial}{\partial x^i} - \xi_j \frac{\partial K^j}{\partial x^i} \frac{\partial}{\partial \xi_i}. \quad (6.2)$$

³This terminology is motivated by the Hamiltonian description of a charged particle moving on a manifold, where the canonical symplectic structure on the cotangent bundle needs to be modified by a pull-back of a closed two-form (magnetic field) from the base manifold. In our case the two-form is the skew-symmetric part of the Schouten tensor, and the inverse of the Ricci scalar plays a role of electric charge. This magnetic term can always be set to zero by an appropriate choice of a connection in a projective class - here we find it convenient not to make any choices at this stage.

This vector field is some-times referred to as the complete lift [35].

Proposition 6.1. *Let K be an affine vector field for a connection ∇ on $U \subset N$. Then its complete lift (6.2) is a Killing vector field for the Walker lift (4.5).*

Proof. Consider the one-parameter group of transformations generated by the vector field (6.2)

$$x^i \longrightarrow x^i + \epsilon K^i + O(\epsilon^2), \quad \xi_i \longrightarrow \xi_i - \epsilon \xi_j \frac{\partial K^j}{\partial x^i} + O(\epsilon^2).$$

This yields

$$\begin{aligned} g &\longrightarrow g + \epsilon \{ \xi_j K^i dx^j d\xi_i - \xi_i K^j dx^i d\xi_j - (\xi_j \xi_i \xi_k K^j) dx^i dx^k \\ &\quad - 2\Gamma_{ik}^j \xi_j (\xi_m K^i) dx^k dx^m + \Gamma_{ik}^j \xi_m \xi_j K^m dx^i dx^k - K^m (\xi_m \Gamma_{ik}^j) \xi_j dx^i dx^k \} + O(\epsilon^2) \\ &= g - \epsilon \xi_k \mathcal{L}_K(\Gamma_{ij}^k) dx^i \odot dx^j + O(\epsilon^2). \end{aligned}$$

Therefore

$$\mathcal{L}_{\tilde{K}} g = -\xi_k \mathcal{L}_K(\Gamma_{ij}^k) dx^i \odot dx^j, \quad (6.3)$$

and the result follows. \square

Recall that a vector field is projective if its flow maps unparametrised geodesics of an affine connection ∇ to unparametrised geodesics. This happens if

$$\mathcal{L}_K \Gamma_{ij}^k = \delta_i^k \rho_j + \delta_j^k \rho_i \quad (6.4)$$

for some one-form ρ_i .

Proposition 6.2. *Let K be a projective vector field with $\rho_i = \nabla_i f$. Then*

$$K - \xi_j \frac{\partial K^j}{\partial x^i} \frac{\partial}{\partial \xi_i} + f \xi_i \frac{\partial}{\partial \xi_i} \quad (6.5)$$

is a conformal Killing vector field for the Walker metric (4.5).

Proof. The proof is similar to that of Proposition (6.1). The one-parameter group of transformation generated by (6.5) is

$$x^i \longrightarrow x^i + \epsilon K^i + O(\epsilon^2), \quad \xi_i \longrightarrow \xi_i - \epsilon \xi_j \frac{\partial K^j}{\partial x^i} - \epsilon f \xi_i + O(\epsilon^2),$$

which gives

$$g \longrightarrow g - \epsilon \xi_k \mathcal{L}_K(\Gamma_{ij}^k) dx^i \odot dx^j - \xi_k dx^k \odot df + \epsilon f g + O(\epsilon^2).$$

This does not change the conformal class iff K satisfies (6.4) with $\rho_i = \nabla_i f$. \square

Finally we give the main result of this Section, and establish a one-to-one correspondence between projective vector fields on $(N, [\nabla])$, and Killing vector fields on the Einstein lift on M .

Theorem 6.3. *Let K be a projective vector field on $(U, [\nabla])$, where $U \subset N$. Then*

$$\mathcal{K} := K - \xi_j \frac{\partial K^j}{\partial x^i} \frac{\partial}{\partial \xi_i} + \frac{1}{\Lambda} \rho_i \frac{\partial}{\partial \xi_i} \quad (6.6)$$

is a Killing vector for the Einstein Riemannian extension (1.1) which is symplectic with respect to the charged symplectic form (1.2). Conversely, any Killing vector field for (1.1) is a lift (6.6) from N of some projective vector field.

Proof. The integrability conditions for (6.4) are [36] (note however that that our sign conventions for the Schouten tensor differ from that in [36], so the sign of the RHS of (6.7) is opposite to what is given in [36])

$$\mathcal{L}_K P_{ij} = -\nabla_i \rho_j. \quad (6.7)$$

Rewrite the Einstein metric (1.1) as

$$g = g_W + \frac{1}{\Lambda} P_S + \Lambda \theta \odot \theta,$$

where g_W is the Walker metric (4.5), P_S is the symmetrised part of the Schouten tensor, and finally $\theta = \xi_i dx^i$. We shall also write $\mathcal{K} = \tilde{K} + K_\rho$, where \tilde{K} is the complete lift (6.2) and $K_\rho := \rho_i \partial / \partial \xi_i$. Then

$$\begin{aligned} \mathcal{L}_K g &= \mathcal{L}_{\tilde{K}} g_W + \Lambda \mathcal{L}_{\tilde{K}} \theta \odot \theta + \frac{1}{\Lambda} \mathcal{L}_K P_S + \frac{1}{\Lambda} \mathcal{L}_{K_\rho} g_W + \mathcal{L}_{K_\rho} (\theta \odot \theta) \\ &= -\xi_k \mathcal{L}_K (\Gamma_{ij}^k) dx^i \odot dx^j + 0 - \frac{1}{\Lambda} (\nabla_i \rho_j) dx^i \odot dx^j + \frac{1}{\Lambda} dx^i \odot d\rho_i \\ &\quad - \frac{1}{\Lambda} \Gamma_{ij}^k \rho_k dx^i \odot dx^j + (\rho_i dx^i) \odot (\xi_j dx^j) = 0, \end{aligned}$$

where we have used (6.3), (6.4) and (6.7).

Now verify the symplectic condition

$$\begin{aligned} \mathcal{L}_K \Omega &= \mathcal{L}_{\tilde{K}} (d\xi_i \wedge dx^i) + \frac{1}{\Lambda} \mathcal{L}_{K_\rho} (d\xi_i \wedge dx^i) + \frac{1}{\Lambda} \mathcal{L}_K (P_{ij} dx^i \wedge dx^j) \\ &= \frac{1}{\Lambda} (d\rho_i \wedge dx^i - d\rho_i \wedge dx^i) = 0 \end{aligned}$$

as the complete lift \tilde{K} is Hamiltonian with respect to $d\xi_i \wedge dx^i$, and we have used the skew part of the integrability conditions (6.7).

To prove the converse, consider a general vector field $\mathcal{K} = K^i \partial / \partial x^i + Q_i \partial / \partial \xi_i$ on M , and impose the Killing equations. The $d\xi_i \odot d\xi_j$ components of these equations imply that $K^j = K^j(x^1, x^2)$. The $d\xi_i \odot dx^j$ components yield the general form (6.6), where ρ_i are some unspecified functions on N . Finally the $dx^i \odot dx^j$ components imply that the vector field $K^i \partial / \partial x^i$ on N is projective. \square

7. EXAMPLES

7.1. Homogeneous model $M = \mathrm{SL}(3, \mathbb{R})/\mathrm{GL}(2, \mathbb{R})$. Consider the flat projective structure on $(N = \mathbb{RP}^2, [\nabla])$, and choose $\Gamma_{ij}^k = 0$. The resulting four manifold is the complement of an \mathbb{RP}^1 sub-bundle in the projective tractor bundle of \mathbb{RP}^2 which can be identified with $M = \mathrm{SL}(3, \mathbb{R})/\mathrm{GL}(2, \mathbb{R})$. We shall establish this result in arbitrary dimension. Consider $N = \mathbb{RP}^n$, with its flat projective structure, and an $\mathrm{SL}(n+1)$ action on the projective tractor bundle $\mathbb{P}(E)$ minus the diagonal

$$\mathrm{SL}(n+1) : \mathbb{R}^{n+1} \times \mathbb{R}_{n+1}/\Delta \longrightarrow \mathbb{R}^{n+1} \times \mathbb{R}_{n+1}/\Delta$$

where the ‘diagonal’ Δ consists of all incident pairs of vectors $[v] \in \mathbb{R}^{n+1}$ and forms $[f] \in \mathbb{R}_{n+1}$ s.t. the corresponding point $v \in \mathbb{RP}^n$ belongs to the hyperplane $f \in \mathbb{RP}_n$. This action is simply $(v, f) \rightarrow (Av, fA^{-1})$. It is transitive, and clearly a subgroup stabilising a pair (point, hyperplane) is $\mathrm{GL}(n)$ which sits in $\mathrm{SL}(n+1)$ as a lower diagonal block.

To finish the proof we need to argue that $\mathbb{R}^{n+1} \times \mathbb{R}_{n+1}/\Delta$ projects down to a complement of an \mathbb{RP}_{n-1} sub-bundle in $\mathbb{P}(E)$. This sub-bundle is just $\mathbb{P}(T^*N)$ and it has an injection into $\mathbb{P}(E)$ given by $f \rightarrow (0, f)$. A point in N with homogeneous coordinates $[1, 0, \dots, 0]$ (corresponding to our choice of an affine chart) is not incident with any tractor in $\mathbb{P}(E)/\mathbb{RP}_{n-1}$, so removing a diagonal is equivalent to looking at the complement of this sub-bundle.

The Einstein metric (5.3) on this manifold admits a Kerr-Schild form

$$g = d\xi_i \odot dx^i + \Lambda(\xi_j dx^j)^2 \quad (7.1)$$

with eight dimensional isometry group $\mathrm{SL}(3, \mathbb{R})$ [12] in agreement with Theorem 6.3). This metric is a neutral signature analog of the Fubini–Study metric on \mathbb{CP}^2 . Both metrics arise as different real forms of $\mathrm{SL}(3, \mathbb{C})/\mathrm{GL}(2, \mathbb{C})$. The limit $\Lambda = 0$ in (7.1) gives the flat metric.

7.2. Ricci–flat limits. Motivated by the previous example let us now consider the general case of projective structures which admit a connection with skew-symmetric Schouten tensor. In this case one can always choose local coordinates on N and a connection $\nabla \in [\nabla]$ such that [39]

$$\Gamma_{11}^1 = -\frac{\partial f}{\partial x^1}, \quad \Gamma_{22}^2 = \frac{\partial f}{\partial x^2},$$

where $f : N \rightarrow \mathbb{R}$ is an arbitrary function, and all other components of ∇ vanish⁴. In this case

$$P = \frac{1}{3} \frac{\partial^2 f}{\partial x^1 \partial x^2} dx^1 \wedge dx^2,$$

⁴An alternative characterisation of the corresponding projective structures is that they arise from second-order ODEs point equivalent to derivatives of first order ODEs [17]. These projective structures were further characterised in [32] and [24].

and the metric is given by

$$g = d\xi_i \odot dx^i + \xi_1 \frac{\partial f}{\partial x^1} (dx^1)^2 - \xi_2 \frac{\partial f}{\partial x^2} (dx^2)^2 + \Lambda (\xi_j dx^j)^2. \quad (7.2)$$

Setting $\Lambda = 0$ gives an ASD Ricci-flat metric which has a form of the Walker lift (4.5) and has appeared in the work of Derdzinski [14].

7.3. Cohomogeneity-one examples. The dimension of the Lie algebra \mathfrak{g} of projective vector fields for a given projective structure on a surface N can be 8, 3, 2, 1 or 0 (see [27], and also [33, 5, 16]). If the dimension is maximal and equal to 8 then $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$, and the projective structure is flat. We have shown that in this case the resulting metric (1.1) is given by (7.1), and admits 8 Killing vectors in agreement with Theorem 6.3. We shall now consider the submaximal case, where $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$. There are two one-parameter families of non-flat projective structures with this symmetry. Their unparametrised geodesics are integral curves of a second order ODE

$$y'' = c(xy' - y)^3,$$

where $c \neq 0$. We compare this to the general second order ODE defining a projective structure (see e.g. [6])

$$y'' = \Gamma_{22}^1 (y')^3 + (2\Gamma_{12}^1 - \Gamma_{22}^2) (y')^2 + (\Gamma_{11}^1 - 2\Gamma_{12}^2) y' - \Gamma_{11}^2, \quad (7.3)$$

and chose the representative connection ∇ by

$$\Gamma_{11}^1 = -\Gamma_{12}^2 = -\Gamma_{21}^2 = cxy^2, \quad \Gamma_{22}^2 = -\Gamma_{21}^1 = -\Gamma_{12}^1 = cx^2y, \quad \Gamma_{22}^1 = cx^3, \quad \Gamma_{11}^2 = cy^3.$$

The corresponding ASD Einstein metric (1.1) is

$$g = d\xi_i \odot dx^i + \Lambda (\xi_i dx^i)^2 + \frac{4c}{\Lambda} (x^2 dx^1 - x^1 dx^2)^2 - \Gamma_{ij}^k dx^i \odot dx^j, \quad (7.4)$$

where $x^i = (x, y)$. This metric admits a three-dimensional isometry group $SL(2, \mathbb{R})$ generated by left-invariant vector fields K_α , $\alpha = 1, 2, 3$ given by

$$K_1 = x^1 \frac{\partial}{\partial x^1} - x^2 \frac{\partial}{\partial x^2} - \xi_1 \frac{\partial}{\partial \xi_1} + \xi_2 \frac{\partial}{\partial \xi_2}, \quad K_2 = 2x^1 \frac{\partial}{\partial x^2} - 2\xi_2 \frac{\partial}{\partial \xi_1}, \quad K_3 = 2\xi_1 \frac{\partial}{\partial \xi_2} - 2x^2 \frac{\partial}{\partial x^1},$$

and acting on $M = \mathbb{R} \times SL(2, \mathbb{R})$ with three-dimensional orbits. We shall use an invariant coordinate r given by $r^2 \equiv (x^1 \xi_1 + x^2 \xi_2)$ which is constant on the orbits. Let σ^α be right-invariant one-forms on $SL(2, \mathbb{R})$ such that

$$\mathcal{L}_{K_\alpha} \sigma^\beta = 0, \quad \forall \alpha, \beta, \quad \text{and} \quad d\sigma^1 + 2\sigma^2 \wedge \sigma^3 = 0, \quad d\sigma^2 + \sigma^2 \wedge \sigma^1 = 0, \quad d\sigma^3 - \sigma^3 \wedge \sigma^1 = 0.$$

There is some freedom, measured by functions of r , in choosing these one-forms. If we chose $\Lambda < 0$, and take

$$\sigma^1 = \frac{\xi_i dx^i - x^i d\xi_i}{r^2} + \frac{2\Lambda r dr}{\Lambda r^2 - 1}, \quad \sigma^2 = \frac{\Lambda r^2 - 1}{r^2} (x^1 dx^2 - x^2 dx^1), \quad \sigma^3 = \frac{\xi_1 d\xi_2 - \xi_2 d\xi_1}{r^2 (\Lambda r^2 - 1)}$$

then the metric (7.4) takes the form

$$g = \frac{dr^2}{1 - \Lambda r^2} - \frac{1}{4}r^2(1 - \Lambda r^2)(\sigma^1)^2 - \frac{c}{\Lambda} \frac{(\Lambda r^2 - 4)r^4}{(\Lambda r^2 - 1)^2}(\sigma^2)^2 + r^2\sigma^2 \odot \sigma^3, \quad \Lambda < 0. \quad (7.5)$$

This metric appears to be singular when $r = 0$, but calculating the invariant norm of the Weyl curvature we find $|C|^2 = 96\Lambda^2$, which is regular. In fact near $r = 0$ the metric (7.5) approaches the space of constant curvature which is a neutral signature analogue of the hyperbolic space. To exhibit this space in a standard form we neglect the small terms involving r^4 , and set $r = 2R/(1 + \Lambda R^2)$. Then, near $R = 0$, the metric (7.5) becomes

$$g \sim \frac{4}{(1 + \Lambda R^2)^2} \left(dR^2 - \frac{R^2}{4} \left((\sigma^1)^2 - 4\sigma^2 \odot \sigma^3 \right) \right).$$

To this end, we note a curious Ricci-flat limit of (7.5). Setting $c = m\Lambda$, and taking the limit $\Lambda \rightarrow 0$ yields a Ricci-flat metric with 9-dimensional group of conformal isometries

$$g = d\xi_i \odot dx^i + 4m(x^2 dx^1 - x^1 dx^2)^2.$$

This is a submaximal metric of neutral signature [12, 25]: if the dimension of the conformal isometry algebra \mathfrak{g} exceeds 9, then $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{R})$, and the metric is conformally flat.

APPENDIX A. THE CONSTRUCTION FOR HIGHER DIMENSIONS

Of course, the definition of a projective structure makes sense in higher dimensions as well and hence it is natural to ask if the construction described in the main body of this article carries over to higher dimensions. Here we briefly show that this is indeed the case.

As usual, let $\mathrm{PGL}(n+1, \mathbb{R})$ denote the quotient of the general linear group $\mathrm{GL}(n+1, \mathbb{R})$ by its center Z , so that

$$\mathrm{PGL}(n+1, \mathbb{R}) \simeq \begin{cases} \mathrm{SL}(n+1, \mathbb{R}) & n \text{ even,} \\ \mathrm{SL}_{\pm}(n+1, \mathbb{R})/\{\pm I_{n+1}\} & n \text{ odd,} \end{cases}$$

where $\mathrm{SL}_{\pm}(n+1, \mathbb{R})$ denotes the group of real $(n+1)$ -by- $(n+1)$ matrices with determinant ± 1 .

The projective linear group acts from the left on $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^*$ by matrix multiplication. The stabiliser subgroup of the line spanned by ${}^t(1 \ 0 \ \dots \ 0)$ will be denoted by $G \subset \mathrm{PGL}(n+1, \mathbb{R})$. The elements of G are matrices of the form

$$\begin{pmatrix} \det a^{-1} & b \\ 0 & a \end{pmatrix}$$

for n even and

$$\begin{bmatrix} \pm \det a^{-1} & b \\ 0 & a \end{bmatrix}$$

for n odd, where $b \in \mathbb{R}_n$ and $a \in \mathrm{GL}(n, \mathbb{R})$. Here, the square brackets indicate that the matrix is only well defined up to an overall sign.

Cartan's construction carries over to higher dimensions so that we canonically obtain a Cartan geometry $(\pi : B_{[\nabla]} \rightarrow N, \theta)$ of type $(\mathrm{PGL}(n+1, \mathbb{R}), G)$ for every projective structure $[\nabla]$ on a smooth n -manifold N . Again, we write

$$\theta = \begin{pmatrix} -\mathrm{tr} \phi & \eta \\ \omega & \phi \end{pmatrix}$$

for an \mathbb{R}_n -valued 1-form η , an \mathbb{R}^n -valued 1-form ω and a $\mathfrak{gl}(n, \mathbb{R})$ -valued 1-form ϕ . The curvature 2-form Θ satisfies

$$\Theta = d\theta + \theta \wedge \theta = \begin{pmatrix} 0 & L(\omega \wedge \omega) \\ 0 & W(\omega \wedge \omega) \end{pmatrix},$$

for smooth curvature functions

$$L : B_{[\nabla]} \rightarrow \mathrm{Hom}(\mathbb{R}^n \wedge \mathbb{R}^n, \mathbb{R}_n)$$

and

$$W : B_{[\nabla]} \rightarrow \mathrm{Hom}(\mathbb{R}^n \wedge \mathbb{R}^n, \mathbb{R}_n \otimes \mathbb{R}^n).$$

The function W represents the *Weyl projective curvature tensor* of $(N, [\nabla])$ and the function L represents the *projective Cotton tensor* of $(N, [\nabla])$. Note that we also have the Bianchi-identity

$$d\Theta = \Theta \wedge \theta - \theta \wedge \Theta,$$

the algebraic part of which reads

$$0 = L(\omega \wedge \omega) \wedge \omega \quad \text{and} \quad 0 = W(\omega \wedge \omega) \wedge \omega. \tag{A.1}$$

We have a Lie group embedding defined by

$$\chi : \mathrm{GL}(n, \mathbb{R}) \rightarrow G, \quad a \mapsto \begin{pmatrix} \det a^{-1} & 0 \\ 0 & a \end{pmatrix},$$

for n even and defined by

$$\chi : \mathrm{GL}(n, \mathbb{R}) \rightarrow G, \quad a \mapsto \begin{bmatrix} |\det a^{-1}| & 0 \\ 0 & a \end{bmatrix},$$

for n odd.

Recall that θ satisfies the equivariance property

$$R_g^* \theta = \mathrm{Ad}(g^{-1}) \circ \theta,$$

for all $g \in G$, where Ad denotes the adjoint representation of G . Identifying $\mathrm{GL}(n, \mathbb{R})$ with its image under χ , the equivariance property of θ implies that the tensor field $\eta \otimes \omega = \eta_i \otimes \omega^i$ is invariant under the $\mathrm{GL}(n, \mathbb{R})$ right action. Furthermore, since ω and η are both semi-basic for the quotient projection $B_{[\nabla]} \rightarrow B_{[\nabla]}/\mathrm{GL}(n, \mathbb{R})$, it follows that the smooth $2n$ -manifold $M = B_{[\nabla]}/\mathrm{GL}(n, \mathbb{R})$ carries a unique signature (n, n) metric g and a unique non-degenerate 2-form Ω having the property that g pulls back to $B_{[\nabla]}$ to

be the symmetric part of $\eta \otimes \omega$ and Ω pulls back to $B_{[\nabla]}$ to be the anti-symmetric part of $\eta \otimes \omega$. Moreover, we compute

$$\begin{aligned} 0 &= d(\eta \wedge \omega) = d\eta \wedge \omega - \eta \wedge d\omega = [-\eta \wedge (\phi + \text{Id tr } \phi) + L(\omega \wedge \omega)] \wedge \omega \\ &\quad - \eta \wedge [-(\phi + \text{Id tr } \phi) \wedge \omega] \\ &= L(\omega \wedge \omega) \wedge \omega, \end{aligned}$$

where we used (A.1). It follows that Ω is symplectic.

We leave it to the interested reader to check that the pair (g, Ω) defines again a bi-Lagrangian structure on M whose symmetry vector fields are in one-to-one correspondence with the symmetry vector fields of $(N, [\nabla])$. Moreover, we may introduce local coordinates on M so that g and Ω take the form (3.11). In particular, the metric g is still Einstein with non-zero scalar curvature, as can be verified by direct computation.

REFERENCES

- [1] Atiyah, M. F. (1970) Vector fields on manifolds. *Arbeitsgemeinschaft für Forschung des Landes Nordrhein Westfalen 200*, 7–24. [18](#)
- [2] Bailey, T. N., Eastwood, M. G. and Gover, A. R. (1994) Thomas’s structure bundle for conformal, projective and related structures. *Rocky Mountain J. Math.* **24**, 1191–1217. [5](#), [16](#)
- [3] Brozos-Vázquez, M., García-Río, E., Gilkey, P., Nikčević, S. and Vázquez-Lorenzo, R. (2009) *The Geometry of Walker Manifolds*. Synthesis Lectures on Mathematics and Statistics. Morgan and Claypool. [15](#), [18](#)
- [4] Bryant, R. L. (2001) Bochner-Kähler metrics. *J. Amer. Math. Soc.* **14**, 623–715. [9](#)
- [5] Bryant, R. L., Manno, G. and Matveev, V. (2008) A solution of a problem of Sophus Lie: normal forms of two-dimensional metrics admitting two projective vector fields. *Math. Ann.* **340**, 437. [23](#)
- [6] Bryant, R. L., Dunajski, M., and Eastwood, M. G. (2009) Metrisability of two-dimensional projective structures, *J. Differential Geometry* **83**, 465–499. [15](#), [23](#)
- [7] Calderbank, D. M. J. (2014) Selfdual 4-manifolds, projective structures, and the Dunajski-West construction. *SIGMA* **10**, 034. [1](#), [13](#), [14](#), [17](#)
- [8] Calderbank, D. M. J. (2014) Integrable Background Geometries. *SIGMA* **10**, 035. [13](#)
- [9] Čap, A. and Slovák, J. (2009) *Parabolic Geometries I: Background and General Theory*, American Mathematical Society 2009. [12](#)
- [10] Cartan, E. (1924) Sur les variétés à connexion projective. *Bull. Soc. Math. France.* **52**, 205–241. [4](#)
- [11] Calvino-Louzao, E., García-Río, E., Gilkey, P. and Vázquez-Lorenzo, R. (2009) The geometry of modified Riemannian extensions. *Proceedings of the Royal Society* **A465**. [17](#)
- [12] Casey, S., Dunajski, M. and Tod, K. P. (2013) Twistor geometry of a pair of second order ODEs *Comm. Math. Phys.* **321**, 681–701. [22](#), [24](#)
- [13] Chudecki, A. and Przanowski, M. (2008) From hyperheavenly spaces to Walker and Osserman spaces: I *Class.Quant.Grav.* **25** 145010. [18](#)
- [14] Derdziński, A. (2008) Connections with skew-symmetric Ricci tensor on surfaces. *Results Math.* **52**, 223–245. [23](#)
- [15] Derdziński, A. (2009) Non-Walker Self-Dual Neutral Einstein Four-Manifolds of Petrov Type III *Journal of Geometric Analysis*, **19** 301–357 [17](#)
- [16] Dumitrescu, S. and Guillot, A. (2013) Quasihomogeneous analytic affine connections on surfaces. *J. Topol. Anal.* **5** 491. [23](#)

- [17] Dunajski, M., and West, S. (2007) Anti-self-dual conformal structures from projective structures. *Comm. Math. Phys.* **272**, 85–118. [14](#), [22](#)
- [18] Dunajski, M. and Tod, K. P. (2010) Four Dimensional Metrics Conformal to Kähler, *Math. Proc. Camb. Phil. Soc.* **148**, 485–503. [15](#)
- [19] Eastwood, M. and Matveev, V. S. (2007) Metric connections in projective differential geometry. In *Symmetries and Overdetermined Systems of Partial Differential Equations*, IMA Vol. Math. Appl., 144. [4](#)
- [20] Hammerl, M. and Sagerschnig, K. (2011) A non-normal Fefferman-type construction of split-signature conformal structures admitting twistor spinors. [arXiv:1109.4231](#). [15](#)
- [21] Hammerl, M., Sagerschnig, K., Silhan, J., Taghavi-Chabert, A. and Zadnik, V. A non-normal Fefferman-type construction of split-signature conformal structures from projective structures. *In preparation*. [15](#)
- [22] Hirzebruch, F. and Hopf, H. (1958) Felder von Flächenelementen in 4-dimensionalen Mannigfaltigkeiten. *Math. Ann.* **136**, 156–172. [18](#)
- [23] Kobayashi, S. and Nagano, T. (1964) On projective connections, *J. Math. Mech.* **13**, 215–235. [12](#)
- [24] Kryński, W. (2014) Webs and projective structures on a plane. *Diff. Geom. Appl.* **37**, 133. [22](#)
- [25] Kruglikov, B. and The, D. (2014) The gap phenomenon in parabolic geometries. *J. reine angew. Math.* [24](#)
- [26] Libermann, P. (1954) Sur le problème d'équivalence de certaines structures infinitésimales, *Ann. Mat. Pura Appl.* (4) **36**, 27–120. [10](#)
- [27] Lie, S. (1882) Untersuchungen über geodätische Kurven, *Math. Ann.* **20**. [23](#)
- [28] Mason, L. J. and Woodhouse, N. M. J. (1996) *Integrability, selfduality, and twistor theory*. Oxford, UK: Clarendon (LMS monographs, new series: 15). [14](#)
- [29] Mettler, T. (2015) Extremal conformal structures on projective surfaces, in preparation. [8](#)
- [30] Nakata, F. (2007) Self-dual Zollfrei conformal structures with alpha-surface foliation. *J. Geom. Phys.* **57**, 2077–2097, [14](#)
- [31] Penrose, R. (1976) Nonlinear gravitons and curved twistor theory, *Gen. Rel. Grav.* **7**, 31–52 [6](#), [18](#)
- [32] Randal, M. (2014) Local obstructions to projective surfaces admitting skew-symmetric Ricci tensor *Jour. Geom. Phys.* **76**, 192. [22](#)
- [33] Romanovskii, Y. R. (1996) Calculation of local symmetries of second-order ordinary differential equations by Cartans equivalence method, *Math. Notes* **60** 56. [23](#)
- [34] Tafel, J. Wojcik, D. (1998) Null Killing vectors and reductions of the self-duality equations, *Non-linearity* **11** 835. [14](#)
- [35] Yano, K. and Ishihara, S. (1973) *Tangent and Cotangent Bundles. Differential Geometry* Marcel Dekker, Inc. New York. [20](#)
- [36] Yano, K. (1955) *The Theory Of Lie Derivatives And Its Applications*. North Holland [21](#)
- [37] Walker, A. G. (1950) Canonical form for a Riemannian space with a parallel field of null planes. *Quart. J. Math. Oxford* **1**, 69 . [18](#)
- [38] Walker, A. G. (1953) Riemann extensions of non-Riemannian spaces. In *Convegno di Geometria Differenziale*. Venice. [15](#)
- [39] Wong, Y. C. (1964) Two dimensional linear connexions with zero torsion and recurrent curvature. *Monatsh. Math.* **68**, 175. [22](#)

DEPARTMENT OF APPLIED MATHEMATICS AND THEORETICAL PHYSICS, UNIVERSITY OF CAMBRIDGE, WILBERFORCE ROAD, CAMBRIDGE CB3 0WA, UK.

E-mail address: `m.dunajski@damtp.cam.ac.uk`

INSTITUT FÜR MATHEMATIK, GOETHE-UNIVERSITÄT FRANKFURT, ROBERT-MAYER-STR. 10, 60325 FRANKFURT, GERMANY.

E-mail address: `mettler@math.uni-frankfurt.de`